Uraniborg Math Handbook

Tycho Brahe et al.
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Ψ*Ψ causaScientia

***Uraniborg* Math Handbook**
Author: Tycho Brahe et al.
Editor: I. L. E. Dreyer and J. Raeder, 1913
Translator: Michael P. McLaughlin, April, 2023

This document produced using LaTeX and TeXShop.

Translator's Preface

The material in this document is a translation of a "handbook" prepared at *Uraniborg*, the observatory of Tycho Brahe, sometime after 1580. The original text may be found in his published (and edited) *Opera Omnia* [2, pp. 283–305]. This handbook is a collection of algorithms for solving various problems in mathematics that arise in the reduction of astronomical data. Most of Tycho's assistants lacked mathematical expertise and had to be taught how to do the necessary calculations. The latter were mainly problems in spherical trigonometry. Very few people, even today, would know the requisite math since it is rarely covered in a high school curriculum. It was no different in the sixteenth century.

The present document presents, first, a translation of the text including many added corrections and clarifications [in brackets]. These, as well as all footnotes, are those of the translator. There is also an appendix describing the algorithms once more but, this time, in modern notation with explanations wherever needed. Without this appendix, the handbook would remain largely inscrutable, even in English.

Tycho Brahe was an astronomer—the greatest observer of all time prior to the invention of the telescope and, arguably, the first real scientist, in the the way that we understand that word. Galileo followed a generation later and then Kepler, Newton, etc. after that. It is always interesting to see how significant achievements were carried out for the first time. This handbook provides some insight into the practice of world-class observational astronomy when it was new.

1 Renaissance Math

You would think that mathematics would be the one language that anyone who had ever studied it would understand. However, it is undoubtedly apparent from the translated text (some of the original shown below) that math was not always what it is today.

DOGMA IV PLANORUM.

Trianguli non Rectanguli, datis duobus lateribus cum Angulo comprehenſo, dantur Anguli, & reliquum latus.

I. Si triangulum habuerit duo æqualia latera, Angulus datus dimidiatus, ablatus â 90, ostendit reliquos Angulos; Latus vero tertium habebis, ſi multiplicaveris latus per dimidium Anguli dati, & dividas per totum: exit dimidium reliqui lateris, quod duplica.

II. Si vero habuerit Triangulum inæqualia latera, & Angulus comprehenſus fuerit acutus, multiplica maius latus datum, & divide per totum: exit inventum II, â quo aufer minus latus, & relinquitur inventum III. Jam duc inventum I in totum, & divide per inventum III: productum in Canone fœcundo dat complementum Anguli, qui lateri maiori opponitur.

Pro latere reliquo, multiplica inventum I per ‖ totum, & divide per ſinum Anguli iam inventi, oppoſiti lateri maiori: exit latus reliquum, quod quærebatur.

At ſi Angulus obtuſus fuerit, duc ſinum complementi obtuſi in maius latus, & divide per totum: exit inventum I. Deinde duc ſinum obtuſi in maius Latus, & divide per totum: exit inventum II. Poſtea adde I inventum dato minori lateri, & habebis inventum III. Jam duc inventum II in totum, & divide per inventum III: exit numerus Anguli, qui adiacet Lateri minori, in Tabulâ fœcundâ inquirendi.

Pro reliquo latere multiplica inventum II per totum, & divide per ſinum Anguli modo inventi, qui adiacet Lateri minori: exit latus III, quod quærebatur.

NB. Est complementum ad 180, ubi Angulus is est obtuſus. Complementum obtuſi habes, ſubtrahendo ab eo 90. Sinum vero obtuſi habes ſubtrahendo eum a 180, & reſidui ſinum accipiendo, prioris complementum ad 90ta. ‖

Figure 1: Plane Algorithm IV

This does not look like math at all. For instance, there are no equations anywhere. The *equals* sign had only recently been "invented" and its use had evidently not yet spread to Denmark. In fact, there is no algebra of any kind; symbolic expressions had not yet been "invented" either. Even figures are scarce. Moreover, what descriptions there are do not seem to make sense. Some are not even correct! We cannot blame Tycho Brahe for this. Most, if not all, of this handbook was written by his assistants. [2, pg. 315]

Clearly, some preliminary background is in order. To address this need, we offer a brief discussion focusing on trigonometry in general as well as the now obsolete technique of prosthaphaeresis then describe the algorithms in modern notation, with examples.

1.1 Background

The notes below are not meant to be a comprehensive historical summary. They are limited to an explanation of the most noticeable ways in which Renaissance math was different from what is standard today. These differences are ubiquitous in the math employed at *Uraniborg*. Other differences will be obvious from the translation.

1.1.1 Trigonometry

Trigonometry was developed primarily because people had a need to do calculations based on measurements, especially in astronomy. Calculating is the thing that most distinguishes trigonometry from geometry. In classical geometry, measuring is forbidden so calculations are never discussed.

In ancient times, the study of astronomy was of great importance, in part because it was the basis of astrology and astrology was important for the same reasons that religion is/was important. Even as late as the seventeenth century, astronomers earned much of their income by casting horoscopes. Tycho Brahe did so repeatedly even though, in later life, he did not really believe that they had any predictive power.

Conceptually, trigonometry starts with the *chord*, the line segment joining the ends of two radii of a circle. Geometry tells you how to construct a chord; trigonometry tells you its length. That is where it all begins.

Around 150 CE, when Claudius Ptolemy wrote the *Almagest*, an astronomy book that remained the ultimate authority on the subject for more than a thousand years, he began by showing how to construct a table of chords—length as a function of angle. After Ptolemy, Hindu mathematicians introduced the notion of the half-chord, today referred to as the *sine* and, in Tycho's day when everything was in Latin, the *sinus* or, sometimes, *sinus rectus* to distinguish it from the *sinus versus* (versine = 1 − cosine).

The key point here is that the sine, a half-chord, was a **length**. As such, its value had units of length. This is in sharp contrast to trigonometry today wherein the sine is not a length but a transcendental function with values and arguments that are necessarily dimensionless (no units). Any modern trigonometry textbook will discuss trigonometric functions with reference to a *unit circle* (i.e., radius = 1) and describe angles, especially within a circle, in terms of *radians* (which seems like a unit but is not). In the sixteenth century, sines still had length (conceptually) so a table of sines contained entries that were listed with units of length based on a circle with a radius chosen for convenience.

What was convenient in those days was simplifying the math as much as possible. This meant, among other things, avoiding fractions[1] so calculations, particularly in astronomy, were carried out using integers as much as possible. In Renaissance trigonometry, the trick employed to avoid fractions was to define sines, cosines, etc. using a standard circle with a very large radius. Astronomers such as Copernicus and Regiomontanus used a standard

[1] Decimals, as used today, were not introduced until the seventeenth century.

circle with a radius of ten thousand.[2] Tycho Brahe used a radius of ten million—seven significant figures, enough precision to count arc-seconds in a complete circle.[3] Therefore, when he looked up the sine of any angle, he found an integer of at most seven digits. For the sine of 90 degrees, the *sinus totus* (or *whole*), he got a value of 10,000,000. As with such tables today, they were used both forwards (to find, for example, the sine of a given angle) or backwards (to find the angle, given its sine). The table also included tables of differences for doing linear interpolation, again forwards or backwards.

The convenience of treating the sine as an integer with units of length had one minor drawback. Sines, cosines, tangents (and their reciprocals) are all **ratios** of lengths and, therefore, dimensionless by definition. Whenever one wished to look up an angle in a table, one had to *pre-multiply* the angle by whatever standard radius that table employed, that is, by *sinus totus*, the *whole* = sin(90). To go backwards from a table entry to an angle[4] required dividing by the *whole* afterwards. This constant flipping of units is found in nearly all of the algorithms in the handbook and makes them appear incomprehensible if you do not understand the rationale underlying the construction of the tables. Finally, these tables were so useful that they were referred to as the *canon foecundus*, the fertile/fruitful canon, another unfamiliar reference found throughout all of Tycho's manuscripts.

Figure 2 shows a page from François Viète's trigonometric tables, published in 1579.[5] The standard radius for his tables is 100,000. The first row shows entries[6] for 0.5 degrees, as follows:

$$\sin(0.5) \quad \cos(0.5)|\tan(0.5) \quad \sec(0.5)|\cot(0.5) \quad \csc(0.5)$$

The angle interval between rows is 1 arc-minute, i.e., 0.5 degrees per page.

1.1.2 Spherical Trigonometry

Triangles are familiar to all but spherical triangles much less so. The latter are drawn on a sphere, not a flat surface, and place a greater strain on the visual imagination. For instance, despite appearances, the two triangles shown in Figure 3 are identical (congruent); they have merely been drawn from different viewpoints (see pg. 27).

Spherical trigonometry, trigonometry with spherical triangles, has been well-known for nearly two thousand years because it is an essential tool in astronomy. [1] Almost all of the computations carried out at *Uraniborg* involved spherical trigonometry, usually finding the side of a spherical triangle given other sides and angles. As we shall see, the side of a spherical triangle is measured in degrees (or radians) since the length of such a side is proportional[7] to the central angle which it subtends.

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[2] Ptolemy used a radius of sixty since much of his math was sexagesimal (base-60).

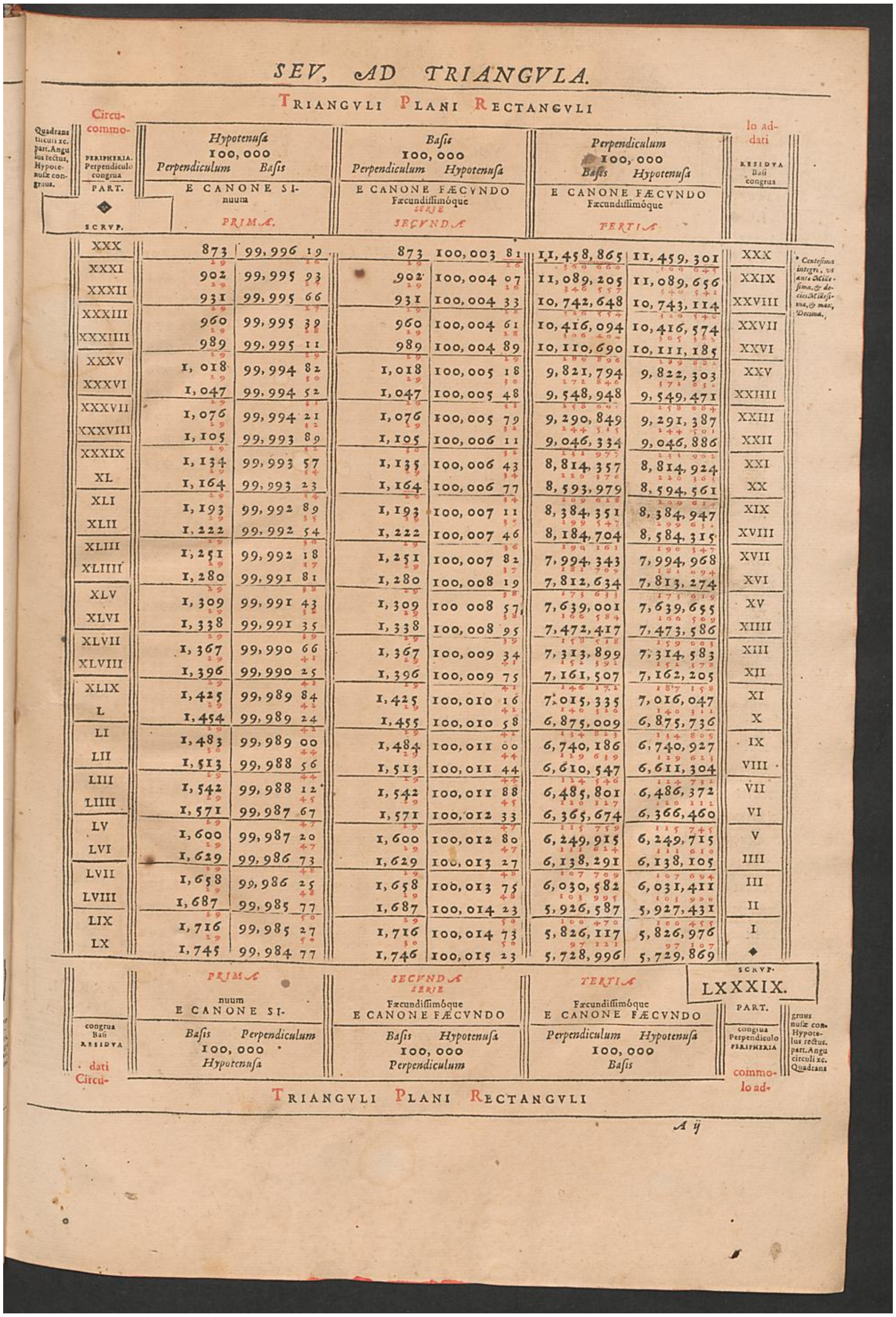
[3] This was essentially that utilized by Regiomontanus in 1541. [6]

[4] i.e., finding an *inverse sine*

[5] retrieved from https://www.e-rara.ch.

[6] some with extra digits, first-order differences shown in red

[7] or equal, in the case of a unit sphere



SEV, AD TRIANGVLA.

TRIANGVLI PLANI RECTANGVLI

SCRVP.	Prima: Hypotenusa 100,000 — Perpendiculum	Basis	Secunda: Basis 100,000 — Perpendiculum	Hypotenusa	Tertia: Perpendiculum 100,000 — Basis	Hypotenusa	Residua
XXX	873	99,996 19	873	100,003 81	11,458,865	11,459,301	XXX
XXXI	902	99,995 93	902	100,004 07	11,089,205	11,089,656	XXIX
XXXII	931	99,995 66	931	100,004 33	10,742,648	10,743,114	XXVIII
XXXIII	960	99,995 39	960	100,004 61	10,416,094	10,416,574	XXVII
XXXIIII	989	99,995 11	989	100,004 89	10,110,690	10,111,185	XXVI
XXXV	1,018	99,994 82	1,018	100,005 18	9,821,794	9,822,303	XXV
XXXVI	1,047	99,994 52	1,047	100,005 48	9,548,948	9,549,471	XXIIII
XXXVII	1,076	99,994 21	1,076	100,005 79	9,290,849	9,291,387	XXIII
XXXVIII	1,105	99,993 89	1,105	100,006 11	9,046,334	9,046,886	XXII
XXXIX	1,134	99,993 57	1,135	100,006 43	8,814,357	8,814,924	XXI
XL	1,164	99,993 23	1,164	100,006 77	8,593,979	8,594,561	XX
XLI	1,193	99,992 89	1,193	100,007 11	8,384,351	8,384,947	XIX
XLII	1,222	99,992 54	1,222	100,007 46	8,184,704	8,584,315	XVIII
XLIII	1,251	99,992 18	1,251	100,007 82	7,994,343	7,994,968	XVII
XLIIII	1,280	99,991 81	1,280	100,008 19	7,812,634	7,813,274	XVI
XLV	1,309	99,991 43	1,309	100 008 57	7,639,001	7,639,655	XV
XLVI	1,338	99,991 35	1,338	100,008 95	7,472,417	7,473,586	XIIII
XLVII	1,367	99,990 66	1,367	100,009 34	7,313,899	7,314,583	XIII
XLVIII	1,396	99,990 25	1,396	100,009 75	7,161,507	7,162,205	XII
XLIX	1,425	99,989 84	1,425	100,010 16	7,015,335	7,016,047	XI
L	1,454	99,989 24	1,455	100,010 58	6,875,009	6,875,736	X
LI	1,483	99,989 00	1,484	100,011 00	6,740,186	6,740,927	IX
LII	1,513	99,988 56	1,513	100,011 44	6,610,547	6,611,304	VIII
LIII	1,542	99,988 12	1,542	100,011 88	6,485,801	6,486,372	VII
LIIII	1,571	99,987 67	1,571	100,012 33	6,365,674	6,366,460	VI
LV	1,600	99,987 20	1,600	100,012 80	6,249,915	6,249,715	V
LVI	1,629	99,986 73	1,629	100,013 27	6,138,291	6,138,105	IIII
LVII	1,658	99,986 25	1,658	100,013 75	6,030,582	6,031,411	III
LVIII	1,687	99,985 77	1,687	100,014 23	5,926,587	5,927,431	II
LIX	1,716	99,985 27	1,716	100,014 73	5,826,117	5,826,976	I
LX	1,745	99,984 77	1,746	100,015 23	5,728,996	5,729,869	◆

LXXXIX. PART.

TRIANGVLI PLANI RECTANGVLI

A ij

Figure 2: A Page From Viète's Trig Tables (1579) [5]

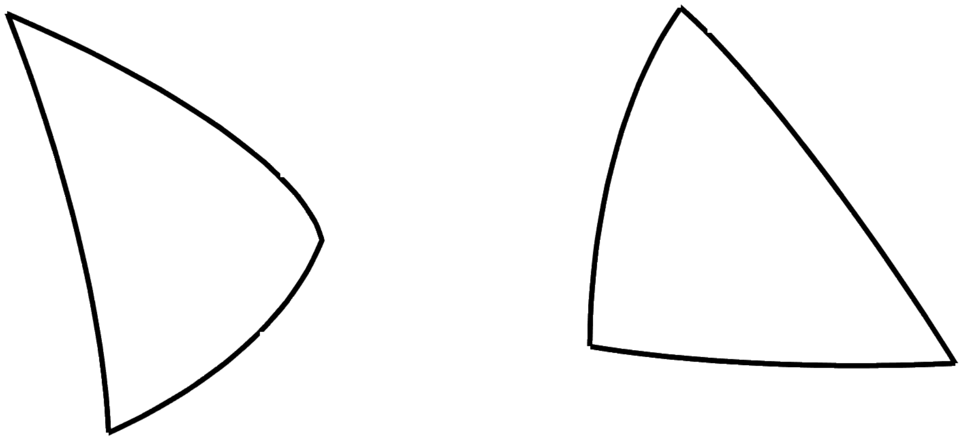


Figure 3: Two Identical Spherical Triangles

1.1.3 Prosthaphaeresis

The technique of prosthaphaeresis ($\pi\varrho o\sigma\vartheta\alpha\phi$) was developed as a method to simplify multiplication, something that became rather tedious when you had to multiply seven-digit numbers over and over again, as was routine at *Uraniborg*. The method worked very well but became obsolete once logarithms were developed, shortly after Tycho's death. In fact, logarithms were used by his former assistant, Johannes Kepler, who incorporated them directly as a novel feature of the Rudolphine Tables.

Prosthaphaeresis took advantage of three trigonometric identities that were known at the time. These identities are listed in Equation 1.

$$\begin{aligned}
\sin(A)\sin(B) &= \frac{\cos(A-B)-\cos(A+B)}{2} \\
\cos(A)\cos(B) &= \frac{\cos(A-B)+\cos(A+B)}{2} \\
\sin(A)\cos(B) &= \frac{\sin(A+B)+\sin(A-B)}{2}
\end{aligned} \tag{1}$$

The left-hand-sides of (1) appear in some basic formulas of spherical trigonometry (see sect. 4.2). These identities enabled anyone doing astronomy to replace a multiplication of large integers with a combination of addition and subtraction. (Dividing by two was trivial.)

It was recognized that the benefit of prosthaphaeresis was not limited to astronomy. *Any* integer, appropriately scaled, could be taken to be the sine of *some* angle that could be found in the body of the fertile canon. You could therefore use (1) to multiply any two numbers provided that you took care of the "decimal point" separately. The latter procedure is very easy and will be familiar to anyone old enough to have used a slide rule.

Using phosthaphaerisis made computations easier for Tycho but more complicated to explain. For instance, consider the algorithm given for finding the side of a right spherical triangle opposite the right angle[8] when the adjacent sides, a and b, are known.

Compare the language of the handbook (cf. page 15):

> *Trianguli Rectanguli, datis duobus lateribus, tertium Recto oppoſitum invenire.*
> *Operatio]* Minus latus, & complementum maioris invicem adde, & aufer, utriuſque reſidui ſinum adde, ſi complementum maioris lateris maius fuerit minore, alias ſubtrahe, & differentiæ cape dimidium, & habebis ſinum complementi lateris reliqui inquirendi. ‖

to the language of today (with $a > b$),

$$\begin{aligned}
\cos(c) &= \frac{\sin[(90-a)+b]+\sin[(90-a)-b]}{2} \\
\text{since,} \quad &= \frac{\sin[90-(a-b)]+\sin[90-(a+b)]}{2} \\
&= \frac{\sin[90-(a-b)]-\sin[(a+b)-90]}{2} \qquad \text{; if necessary} \\
&= \frac{\cos(a-b)+\cos(a+b)}{2} \\
&= \cos(a)\cos(b)
\end{aligned}$$

just to avoid that final multiplication.

1.2 Algorithms

These algorithms were in regular use by Tycho Brahe and his staff from 1580 onwards. The handbook was compiled later.

In the algorithms and the figures, angles (vertices) and other points will be labeled with an uppercase letter and opposite sides in the triangle by the corresponding lowercase letter. Other line segments will be labeled with a lowercase letter or a point pair, e.g., $\overline{AB}$ or $\widehat{AB}$. Known quantities will be shown in red; those sought will be marked with a ✓.

For each algorithm, we shall give a numerical example, including one from Tycho's logs. Here, we shall utilize the algorithms shown in the handbook even if there is a simpler form so that each one matches the translation.[9] When describing spherical triangles, this means using phosthaphaeresis instead of multiplication. We shall also supply a relevant figure, drawn to scale.

[8] almost always, the only one

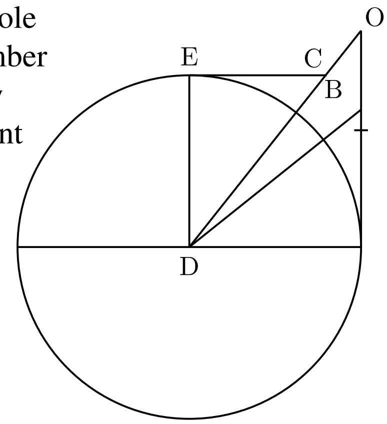
[9] Numerical output shown may have some round-off error.

2 Algorithms for Plane Triangles [P1 – P7]

2.1 Algorithm P1

Given two sides adjacent to the right angle of a right triangle, to find both of the other angles and the remaining side.

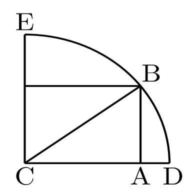
Demonstration of this Algorithm



Angles Multiply the side opposite the angle sought by the whole and divide the result by the remaining side to give a number which, sought in the body of the fertile canon, will show the desired angle. The remaining angle is the complement of 90.

Side To find that which is subtended by the right angle, multiply another side by the whole and divide by the sine of the angle by which the same side is subtended. The desired side, which is opposite to the right angle, is produced.

2.2 Algorithm P2



Given two sides of a right triangle, one adjacent to the right angle and the other opposite the same, to find the angles and the remaining side.

Multiply the side adjacent to the right angle by the whole and divide [the result] by the side subtending the right angle to give the sine of the angle subtended by that side. The third angle is given by the complement of 90 of this arc. If you multiply the sine of this [third angle] by the side subtending the right angle and divide by the whole, it gives the other side adjacent to the right angle that was sought.

2.3 Algorithm P3

Given the other acute angles of a right triangle, along with one side, to find the remaining two sides.

I. If the side subtending the right angle is given, multiply the length of this side by the sine of the angle that is opposite the side being sought and divide by the whole to give the side that was sought.

II. If, however, one of the two sides adjacent to the right angle is given and you wish to measure the side subtending the right angle, multiply the given side by the whole and divide by the sine of the angle opposite to it to give the length of the side opposite the right angle.

III. If you want to find the remaining side adjacent to the right angle, multiply the length of the given side by the sine of the complement [cosine][10] of the angle opposite to it and divide by the sine of the angle itself. This gives the remaining side that was sought.

2.4 Algorithm P4

Given two sides and the included angle of a non-right triangle, to find the [other] angles and the remaining side.

I. If the triangle has two equal sides, the given angle halved, subtracted from 90, gives the remaining [equal] angles. For the third side: If you multiply the [given] side by [the sine of] half the given angle and divide by the whole, it yields half of the remaining side, which is to be doubled.

II. However, if the triangle has unequal sides and the included angle is acute, [multiply the given larger side by the sine of the given angle, and divide by the whole to give finding I.][11] Multiply the larger side by the cosine of the given angle, and divide by the whole yielding finding II from which subtract the smaller side to give finding III. Now, multiply finding I by the whole and divide by finding III. The result, found in the fertile canon, gives the complement [supplement] of the angle which is opposite the larger side.

For the remaining side, multiply finding I by the whole and divide by the sine of the angle just found, opposite the larger side.[12] This gives the remaining side which was sought.

But, should the angle be obtuse, multiply the sine of the complement of the obtuse angle [supplement] by the larger side and divide by the whole to give finding I. Then, multiply the sine of the obtuse angle by the larger side and divide by the whole to give finding II. Afterwords, add finding I to the smaller side to give finding III. Now, multiply finding II by the whole and divide by finding III. This, sought in the fertile canon, gives the value of the angle adjacent to the smaller side.

For the remaining side, multiply finding II by the whole and divide by the sine of the angle just found that is adjacent to the smaller side. This gives the third side which was sought.

N.B. Where the angle is obtuse, the complement [supplement] is from 180. You have the complement [of this supplement] by subtracting 90 from it [the obtuse angle]. Actually, you have the sine of an obtuse angle by subtracting it from 180 and taking the sine of the remainder, as before, the complement of 90.

[10] Hereafter, this will be translated as *cosine* directly.

[11] This definition is absent in the original text.

[12] This angle is obtuse so its sine equals that of its supplement.

2.4.1 A Shorter Solution for this Algorithm Without Dropping a Perpendicular

1. Given an angle (acute or obtuse), [find] the complement of the semicircle, that is 180, [take] half, and find the tangent [of the latter] from the Canon of Tangents.

2. Then, add the smaller given side to the larger and find half the sum which will be finding I.

3. From finding I subtract the smaller side again and the remainder will be finding II.

4. Now the third finding will be the tangent of the complement of the angle previously halved.[13]

Practice

Now, multiply finding II by finding III and divide by finding I to give the tangent of [half of] the difference of the angles which were subtended by the given sides. Therefore, if you subtract this difference found in the Canon of Tangents from the angle previously found and halved, it will be clear which angle was subtended by the smaller of the given sides. Again, if you add this to the same angle found from the complement, the angle subtended by the larger side is revealed. Or else, given two angles in a planar triangle, there exists a third by their complement known as the semicircle.

2.5 Algorithm P5

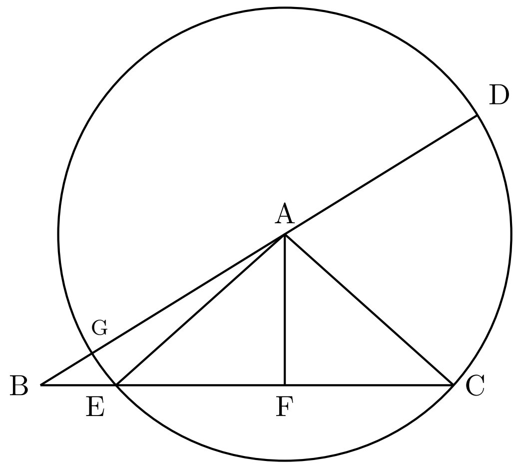
Given a non-right triangle with two sides and an opposite angle, or two angles with a side that is either subtended or a hundred degrees apart, find the remainder from here.

[no text]

[13] i.e., the output of step 1

2.6 Algorithm P6

Given three sides of a non-right triangle, to find all of the angles.



Demonstration of Algorithm P6

I. Equal-sided In a triangle of three equal sides, which they call *isopleurum*, the angles are of 60 parts [here, degrees].

II. Equal-legged In a triangle of two equal sides, which they call *aequicrurium*, multiply half the base by the whole, and divide by one of the equal sides. This gives the sine of the half-angle enclosed by the two equal sides. Therefore, since the rest is the same, the complement of this will indicate the amount of either of the remaining angles.

III. Scalene In a triangle of three unequal sides, take the smallest side and the next-larger side, add them to each other, and subtract them, multiply these two by each other and divide [the product] by the length by the greatest side. This gives finding I. Subtract this from the largest side, halve the remainder and add finding I and you will have finding II. Multiply this second finding by the whole and divide it [the product] by the middle side, that is, nearest the smallest and largest. This gives the complement of the angle opposite the smallest side.

For the rest of the angles, take the half of the remainder previously given before (from the half-completed second finding) and redo [with] the smallest side, as if it were opposite the angle (which is implied). This gives the complement of the angle opposite the middle side. Therefore, the third [angle] will be known.

2.7 Algorithm P7

Knowing the three angles of a triangle, to find each of the sides.

Double the sine of any angle and the length of the side opposite to this angle will result in such parts since the radius of the circle is the whole sine.

[The sides cannot be determined from angles alone since similar triangles have all their angles equal.]

3 Algorithms for Spherical Triangles [S1 – S9]

3.1 Algorithm S1

Given two sides of a right triangle, find the third opposite the right [angle].

Operation:
Add the smaller side and the complement of the larger one to each other, and take away. Add the sine of each result. If the smaller [side] is greater than the complement of the larger, subtract the other[14] and take half of the difference and you will have the cosine of the remaining side sought.

3.2 Algorithm S2

Given two sides of a right triangle, to find whichever angle.

Operation:
If the side opposite the right angle has not been given, this side must first be sought by the previous operation [3.1] and from there proceed in this manner [Sine Rule]: Multiply the sine of the side opposite the angle sought by the whole sine and divide the product by the sine of the side subtending the right angle. The sine of the angle sought is obtained, the arc of which measures the size of that angle.

3.3 Algorithm S3

Given one angle of a right triangle with an adjacent side, to find the other angle opposite the given side.

Operation:
Add and subtract the smaller given and the complement of the larger. If the angle is larger than the adjacent side, add the sines of the results; otherwise, subtract them. Take half of the result to get the cosine of the angle sought, or the sine of the side opposite the given angle when the angle is larger than the adjacent side.

N.B. This applies when a side adjacent to the right angle is given. It should be noted that, if the length of the given side is less than the given angle, both must be added, and taken away, and the sines of the results added and halved to give the cosine of the angle sought. However, when the side is greater than the angle, you can use it, as in the attached example, by subtracting the sines, taking the data itself, not the complement of the larger.[15]

[14] That is, if θ is negative, replace $\sin(\theta)$ with $-\sin(-\theta)$ for consistency with the sine table.
[15] Actually, complements should not be used in either case.

But if both givens exceed the quadrant [90°], then work with the complement [supplement] of each,[16] combining them then halving to find the cosine of the angle sought.

3.4 Algorithm S4

With right triangles, given one angle and any side, to find the rest.

I. If the given side is not opposite the right angle [and is opposite the given angle], then multiply the sine of this side by the whole and divide by the sine of the given angle. This gives the sine of the side subtending the right angle. Algorithm S1 [Cosine Rule] then gives the remaining side, Algorithm S2 the angles.

II. If the given side is opposite the right angle, multiply its sine by the sine of the given angle and divide the product by the whole. This gives the sine of the side opposite the given angle. Find the rest by the first and second algorithms, as before.

If the right angle and the given angle surround the given side, then use the previous [Algorithm S3] for the remaining angle and the following for the remaining sides.

3.5 Algorithm S5

Given the two remaining angles of a right triangle, to find any side.

Multiply the cosine of the non-right angle opposite to the subtending side sought by the whole and divide the product by the sine of the remaining non-right angle. The result is the cosine of the side sought.

For the rest of the sides, multiply the sine of the side just found by the sine of the angle looking at the other side sought, whether a right angle or not, and divide the product by the sine of the angle which the newly found side subtends. This will produce the sine of the desired side.

3.6 Algorithm S6

Given two sides and the included angle of a non-right triangle, to find the remaining side.

Add the smaller side and the complement of the larger, and take away. Add the sines of the results to each other if the supplement of the larger side is greater than that of the smaller side; otherwise take it away when it is smaller [negative argument]. Half of the result will be finding I which, subtracted from the first addend, will give finding II. If the given angle is acute, multiply II by the cosine of the angle; if it obtuse, multiply by the sine of the excess by which it exceeds the quadrant. Divide by the whole and you will

[16] $\sin(180 - \theta) = \sin(\theta)$

have finding III. Add this to finding I if the angle given is acute or subtract from finding I if obtuse. You will have the cosine of the side sought.

An Observation Regarding This Algorithm

If one of the sides surrounding the given angle is greater than the quadrant, the resulting opposite side is less than the quadrant. If the product of multiplication is greater than finding I, subtract finding II multiplied by finding I. The cosine of the side sought is left.

3.7 Algorithm S7

Given two angles about a side of a non-right triangle, to find the remaining angle.

The original text contains two errors: a sign error in the last step and a missing halving operation earlier. Both errors have been corrected in the following description which is, once again, just the Cosine Rule.

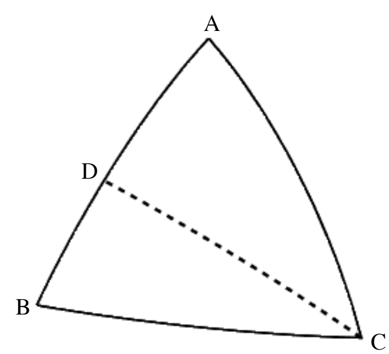
Add the smaller angle and the complement of the larger angle to each other, and take away. Add the sines of the results if the complement of the larger angle exceeds the smaller angle, or subtract if it is less. [Halve the result] to give finding I which, subtracted from the first addend, will give finding II. Multiply [this] by the cosine of the given side then divide by the whole to give finding III. [Subtract finding I from this] and you will get the cosine of the angle sought. Thus, the operation is similar to the previous one.

3.8 Algorithm S8

Given two angles and a side opposite one in a non-right triangle, to find the remaining sides and angle.

This also contains errors and omissions. Changes shown will produce a correct result.

Here, it is necessary to resolve the triangle into two right triangles by drawing a perpendicular from one of the unknown angles, falling inside or outside the triangle, and then proceed using III and IV. For example, let ABC be the triangle, the angles at A and B being known along with with the side $\widehat{AC}$. I want to find the side $\widehat{BA}$. I therefore draw perpendicular $\widehat{CD}$ from unknown angle C to $\widehat{BA}$. Therefore, right triangle CAD has the given angle A and adjacent side $\widehat{AC}$. Therefore, $\widehat{AD}$ will be given by the third [III].



I. Add the smaller to the complement of the larger and take away. Subtract the sines of the results then halve and you will have the cosine of side $\widehat{AD}$. [Need $\widehat{DC}$ first (Sine Rule), then $\widehat{AD}$ (Cosine Rule), then angle ACD, referenced below.]

II. Then, using procedure I [S4 II], find $\widehat{DC}$.

III. Finally, in the other right triangle BDC, for the side $\widehat{BC}$ opposite to the right angle, multiply the sine of the side $\widehat{DC}$ by the whole and divide by the sine of the given angle at B. This gives the sine of side $\widehat{BC}$.

IV. For side $\widehat{BD}$ use procedure I, the smaller side $\widehat{DC}$ and the supplement of the greater $\widehat{BC}$, add to each other and take away. Add the sines of each and halve and you will have the cosine of side $\widehat{BD}$. [Partly unclear. See S4]

Therefore, add the side $\widehat{BD}$ to $\widehat{AD}$ previously found and you will have the whole $\widehat{BA}$ known. Now, for the angle ACB use procedure III, as above, to find angle DCB which must be added to angle ACD previously found and the whole angle ACB which was sought will be found. The operation is similar when the perpendicular falls outside the triangle.

3.9 Algorithm S9

Given three sides of a non-right triangle, to find any angle.

Of two sides surrounding the angle sought, add the smaller side and the supplement [complement] of the larger side to each other, and take away. If the complement of the greater side exceeds the smaller, add the sines of the results; otherwise subtract them. Half of the product is finding I which, subtracted from the first addend, gives finding II. [cf. S6] The third finding is the difference between finding I and the cosine of the remaining side opposite the angle sought. Now, multiply finding III by the whole and divide by finding II. This gives the cosine of the angle sought if it is acute but, if it is obtuse, the arc found must be joined to the quadrant. (This angle is acute when finding I is less than the cosine of the third, opposite side but obtuse when it is greater than the same.)

Various Cases

I. N.B. If the surrounding sides exceed 90 [degrees], take the complement from the south pole.

II. When the greater side exceeds the quadrant, it is necessary to work through its excess of 90. Finding III is then obtained by adding finding I to the cosine of the side subtending the angle sought, not subtracting it as described in the algorithm.

III. If one of the surrounding sides exceeds 90 and the opposite side exceeds 90, add the whole sine to finding I and divide by finding II. This produces the complement of the angle sought. In other words,

IV. If one side surrounding the angle sought exceeds the quadrant and the opposite side does as well, subtract finding I from this sine.

4 Examples

Even in translation, the algorithms in this handbook are not completely transparent. The numerical examples in this section should provide the necessary clarification.

In each, we shall use modern notation and the unit circle. By setting *sinus totus* = 1, we will eliminate any multiplication or division by this factor appearing in the original description.

These examples bring up another issue, not specific to the algorithms. In general, sine tables go from zero to ninety degrees with sines from zero to one. However, in numerical computations, values outside these ranges (obtuse angles, negative sines) often occur. Such quantities are readily handled using complements and supplements of angles, something that must have been routine at *Uraniborg*.

4.1 Plane Algorithms

Algorithm P1

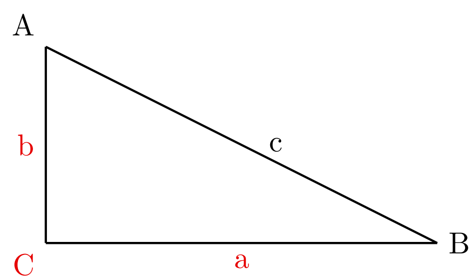
Given two sides adjacent to the right angle of a right triangle, to find both of the other angles and the remaining side.

Angles:

$$I = \frac{a}{b} = \tan(A)$$

$$\checkmark\, A = \tan^{-1}(I)$$

$$\checkmark\, B = 90 - A$$



Side:

$$I = \frac{a}{c} = \sin(A)$$

$$\checkmark\, c = \frac{a}{\sin(A)}$$

Example P1

C = 90 a = 10 b = 5

Angles:

$A = \tan^{-1}(10/5) = 63.4349°$

$B = 90 - A = 26.5651°$

Side:

$c = 10/\sin(A) = 11.1803$

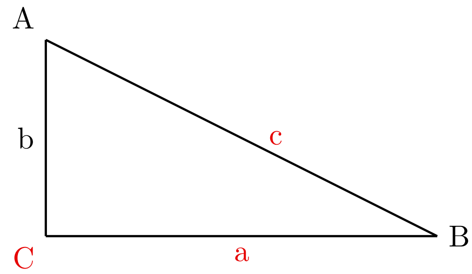
Algorithm P2

Given two sides of a right triangle, one adjacent to the right angle and the other opposite the same, to find the angles and the remaining side.

✓ $A = \sin^{-1}\left(\frac{a}{c}\right)$

✓ $B = 90 - A$

✓ $b = c \sin B$



Example P2

C = 90 a = 10 c = 11.1803

$10/11.1803 = \sin(A) = 0.894427$

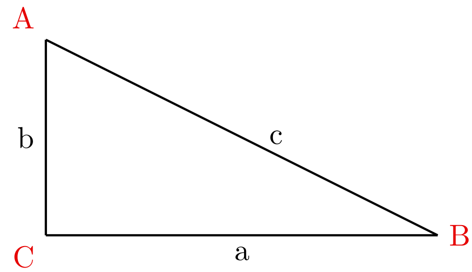
$A = \sin^{-1}(0.894427) = 63.4349°$; from fertile canon

$B = 90 - A = 26.5651°$

$b = 11.1803 \sin(B) = 5$

Algorithm P3

Given the other acute angles of a right triangle, along with one side, to find the remaining two sides.



Given c:

$$\checkmark\, b = c\sin(B)$$

Given a:

$$\checkmark\, c = \frac{a}{\sin(A)}$$

Last side (given a):

$$\checkmark\, b = \frac{a\sin(B)}{\sin(A)} = c\sin(B)$$

Example P3

A = 63.4349° B = 26.5651° C = 90

c = 11.1803

$b = c\sin(B) = 5$

a = 10

$c = \dfrac{a}{\sin(A)} = 11.1803$

Last side (given a)

$b = \dfrac{a\sin(B)}{\sin(A)} = 5$

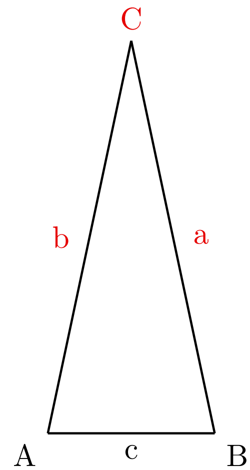
Algorithm P4

Given two sides and the included angle of a non-right triangle, to find the [other] angles and the remaining side.

Isoceles:

$$\checkmark\; A = B = 90 - \frac{C}{2}$$

$$\checkmark\; c = 2\,a\,\sin\left(\frac{C}{2}\right)$$



Example P4 Isoceles

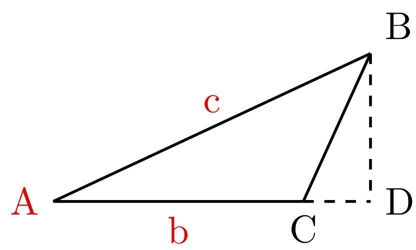
C = 24°   a = b = 10

$A = B = 90 - 24/2 = 78°$

$c = 2\,(10)\,\sin(24/2) = 4.15823$

Acute:

$$I = c\sin(A) = \overline{BD}$$
$$II = c\sin(90 - A) = \overline{AD}$$
$$III = II - b = \overline{CD}$$
$$\angle BCD \equiv \phi = \tan^{-1}\left(\frac{I}{III}\right)$$
$$\checkmark\, C = 180 - \phi$$
$$\checkmark\, B = 180 - A - C$$
$$\checkmark\, a = \frac{I}{\sin(\phi)}$$



Example P4 Acute

A = 25° b = 5 c = 7

$I = 7\sin(25) = 2.95833$

$II = 7\sin(90 - 25) = 6.34415$

$III = II - 5 = 1.34415$

$\phi = \tan^{-1}\left(\frac{I}{III}\right) = 65.5647°$

$C = 180 - \phi = 114.435°$

$B = 180 - A - C = 40.5647°$

$a = \frac{I}{\sin(\phi)} = 3.24938$

Obtuse:

$$I = b\sin(A-90) = \overline{AD}$$

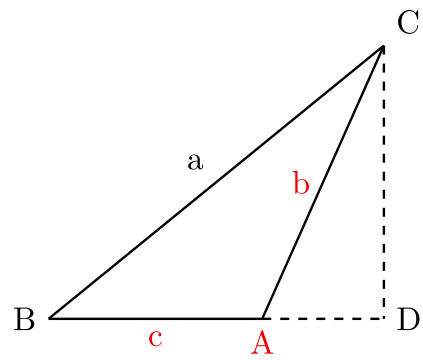
$$II = b\sin(A) = \overline{CD}$$

$$III = c + I = \overline{BD}$$

$$\checkmark\, B = \tan^{-1}\left(\frac{II}{III}\right)$$

$$\checkmark\, C = 180 - A - B$$

$$\checkmark\, a = \frac{II}{\sin(B)}$$



Example P4 Obtuse

A = 114°   b = 7   c = 5

$$I = 7\sin(114-90) = 2.84716$$

$$II = 7\sin(114) = 6.39482$$

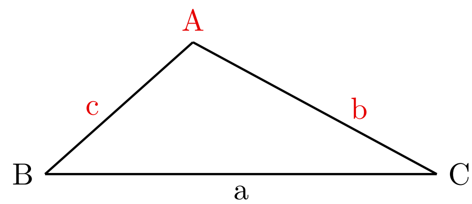
$$III = 5 + I = 7.84716$$

$$B = \tan^{-1}\left(\frac{II}{III}\right) = 39.1773°$$

$$C = 180 - A - B = 26.8227°$$

$$a = \frac{II}{\sin(B)} = 10.1228$$

Alternate(shorter):



$$x = \tan\left(\frac{180 - A}{2}\right)$$

$$I = \frac{b + c}{2}$$

$$II = I - c \quad ; b > c$$

$$III = x$$

$$tanHalfDiff = \frac{II \times III}{I}$$

$$\checkmark\, B = \tan^{-1}(tanHalfDiff) + 90 - \frac{A}{2}$$

$$\checkmark\, C = 180 - A - B$$

$$\checkmark\, a = c\frac{\sin(A)}{\sin(C)} \quad \text{; sine rule}$$

Example P4 Alternate

A = 110°   b = 7   c = 5

$$x = \tan\left(\frac{180 - 110}{2}\right) = 0.700208$$

$$I = \frac{7 + 5}{2} = 6$$

$$II = I - 5 = 1$$

$$III = x = 0.700208$$

$$tanHalfDiff = \frac{II \times III}{I} = 0.116701$$

$$B = \tan^{-1}(tanHalfDiff) + 90 - 110/2 = 41.6564°$$

$$C = 180 - 110 - B = 28.3436°$$

$$a = 5\frac{\sin(110)}{\sin(C)} = 9.89654$$

Algorithm P6

Given three sides of a non-right triangle, to find all of the angles.

Equal-sided:

$$\checkmark\, A = B = C = 60°$$

Equal-legged (a and b):

$$\phi = \sin^{-1}\left(\frac{c}{2b}\right)$$

$$\checkmark\, A = B = 90 - \phi$$

$$\checkmark\, C = 2\phi$$

Scalene:

$$I = (b+a)(b-a)/c$$

$$II = \frac{c-I}{2} + I$$

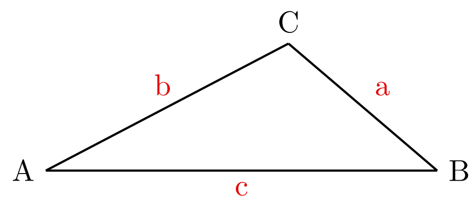
$$x = \sin(90-A) = \frac{II}{b}$$

$$\checkmark\, A = 90 - \sin^{-1}(x)$$

$$y = \sin(90-B) = \frac{c-I}{2a}$$

$$\checkmark\, B = 90 - \sin^{-1}(y)$$

$$\checkmark\, C = 180 - A - B$$



Example P6 (scalene)

a = 5 b = 7 c = 10

$I = (7+5)(7-5)/10 = 2.4$

$II = \frac{10-I}{2} + I = 6.2$

$x = \sin(90-A) = \frac{II}{7} = 0.88571$

$A = 90 - \sin^{-1}(x) = 27.66045°$

$y = \sin(90-B) = \frac{10-I}{10} = 0.76$

$B = 90 - \sin^{-1}(y) = 40.53580°$

$C = 180 - A - B = 111.80375°$

4.2 Spherical Algorithms

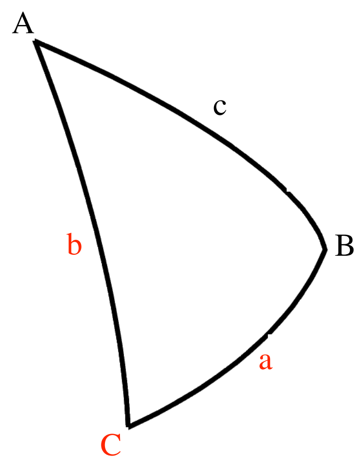
Algorithm S1

Given two sides of a right triangle, find the third opposite the right [angle].

$\angle C = 90°$:

$$x = \cos(a)\cos(b)$$
$$= \frac{\sin[(90-b)+a] + \sin[(90-b)-a]}{2}$$
$$= \frac{\sin[(90-b)+a] - \sin[a-(90-b)]}{2}$$

$$\checkmark\ c = \cos^{-1}(x)$$



Example S1[17]

C = 90° a = 40° b = 45°

$$x = \frac{\sin((90-45)+40) + \sin((90-45)-40)}{2}$$
$$x = \frac{\sin(85) + \sin(5)}{2} = 0.541675$$

$$c = \cos^{-1}(0.541675) = 57.2022°$$

[17] as often, the picture is somewhat deceptive

Algorithm S2

Given two sides of a right triangle, to find whichever angle.

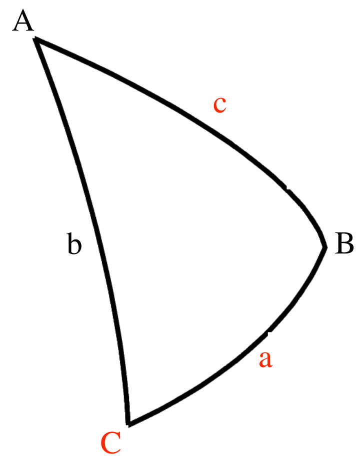
$\angle C = 90°$

Find $\angle A$ using the Sine Rule (after S1, if side c is unknown):

$$\frac{\sin(a)}{\sin(A)} = \frac{\sin(c)}{\sin(90)} = \sin(c)$$

$$x = \sin(A) = \frac{\sin(a)}{\sin(c)}$$

$$\checkmark\ A = \sin^{-1}(x)$$



Example S2

C = 90°   a = 40°   c = 57.2022°

$$x = \frac{\sin(40)}{\sin(57.2022)} = 0.764689$$

$$A = \sin^{-1}(0.764689) = 49.8793°$$

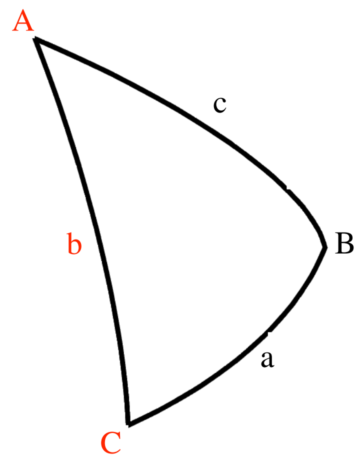
Algorithm S3

Given one angle of a right triangle with an adjacent side, to find the other angle opposite the given side.

$\angle C = 90°$

Find $\angle B$ using the Cosine Rule:

$$x = \sin(A)\cos(b)$$
$$= \frac{\sin(A+b)+\sin(A-b)}{2}$$
$$\checkmark\, B = \cos^{-1}(x)$$



Example S3

C = 90° A = 49.8793° b = 45°

$$x = \frac{\sin(94.8793)+\sin(4.8793)}{2}$$
$$= \frac{\sin(180-94.8793)+\sin(4.8793)}{2}$$
$$= \frac{\sin(85.1207)+\sin(4.8793)}{2} = 0.540717$$

$$B = \cos^{-1}(0.540717) = 57.2676°$$

Were $b > A$, the subtraction mentioned earlier would be used.

Algorithm S4

With right triangles, given one angle and any side, to find the rest.

$\angle C = 90°$ Case I, given A and a:

$$x = \frac{\sin(a)}{\sin(A)}$$

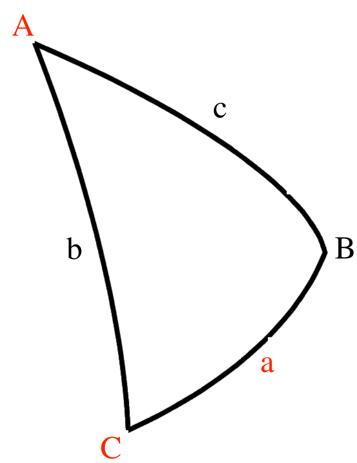
$$\checkmark\, c = \sin^{-1}(x)$$

$$x = \frac{\cos(c)}{\cos(a)} \quad \text{[Cosine Rule]}$$

$$\checkmark\, b = \cos^{-1}(x)$$

$$x = \sin(A)\cos(b) = \frac{\sin(A+b) + \sin(A-b)}{2}$$

$$\checkmark\, B = \cos^{-1}(x)$$



Example S4

C = 90° A = 49.8793° a = 40°

$$x = \frac{\sin(40)}{\sin(49.8793)} = 0.84059$$

$$c = \sin^{-1}(0.84059) = 57.2022°$$

$$x = \frac{\cos(57.2022)}{\cos(40)} = 0.70711$$

$$b = \cos^{-1}(0.70711) = 45°$$

$$x = \frac{\sin(94.8793) + \sin(4.8793)}{2} \quad \text{(see Example S3)}$$

$$= \frac{\sin(180 - 94.8793) + \sin(4.8793)}{2}$$

$$= \frac{\sin(85.1207) + \sin(4.8793)}{2} = 0.54072$$

$$B = \cos^{-1}(0.54072) = 57.2676°$$

Algorithm S5

Given the two remaining angles of a right triangle, to find any side.

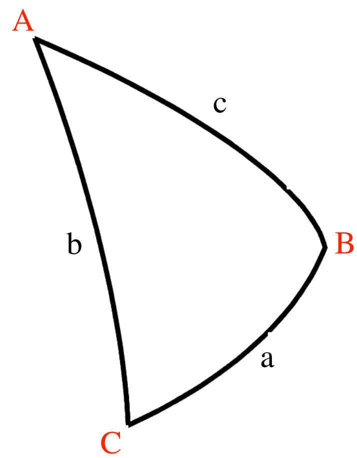
$\angle C = 90°$ Find a and b:

$$x = \frac{\cos(A)}{\sin(B)} \quad \text{[Cosine Rule]}$$

$$\checkmark\, a = \cos^{-1}(x)$$

$$x = \frac{\sin(a)\sin(B)}{\sin(A)} \quad \text{[Sine Rule]}$$

$$\checkmark\, b = \sin^{-1}(x)$$



Example S5

C = 90°   A = 49.8793°   B = 57.2676°

$$x = \frac{\cos(49.8793)}{\sin(57.2676)} = 0.76604$$

$$a = \cos^{-1}(0.76604) = 40°$$

$$x = \frac{\sin(40)\sin(57.2676)}{\sin(49.8793)} = 0.70711$$ (using phosphaeresiss on the numerator)

$b = \sin^{-1}(0.70711) = 45°$ (Find c in the same way.)

Algorithm S6

Given two sides and the included angle of a non-right triangle, to find the remaining side.

Given a, b and C, find c:

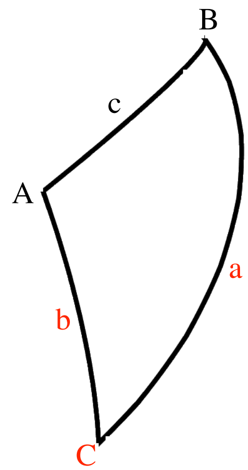
$$I = \cos(a)\cos(b) \quad ;(a > b)$$
$$= \frac{\sin[(90-a)+b] + \sin[(90-a)-b]}{2}$$

$$II = \sin(a)\sin(b)$$
$$= \frac{\sin[(90-a)+b] - \sin[(90-a)-b]}{2}$$

$$III = II \times \cos(C)$$

$$x = I + III$$

$$\checkmark\, c = \cos^{-1}(x)$$



Example S6

a = 81.3508° b = 40° C = 59.6187°

$$I = \frac{\sin[(90-81.3508)+40] + \sin[(90-81.3508)-40]}{2}$$
$$= \frac{\sin[(90-81.3508)+40] - \sin[40-(90-81.3508)]}{2} = 0.115201$$

$$II = \frac{\sin[(90-81.3508)+40] - \sin[(90-81.3508)-40]}{2}$$
$$= \frac{\sin[(90-81.3508)+40] + \sin[40-(90-81.3508)]}{2} = 0.635478$$

$$III = II \times \cos(59.6187) = 0.321394$$

$$x = I + III = 0.436594$$

$$c = \cos^{-1}(0.436594) = 64.1132°$$

Algorithm S7

Given two angles about a side of a non-right triangle, to find the remaining angle.

Given B, C and a, find A:

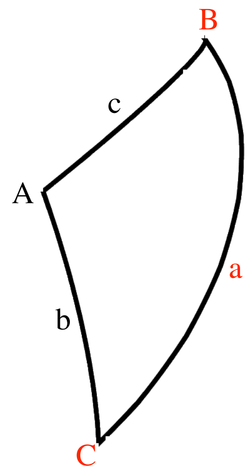
$$I = -\cos(B)\cos(C) \quad ;(C > B)$$
$$= -\frac{\sin[(90-C)+B]+\sin[(90-C)-B]}{2}$$

$$II = \sin(B)\sin(C)$$
$$= \frac{\sin[(90-C)+B]-\sin[(90-C)-B]}{2}$$

$$III = II \times \cos(a)$$

$$x = I + III$$

$$\checkmark\ A = \cos^{-1}(x) = 180 - \cos^{-1}(-x)$$



Example S7

B = 38.0513°   C = 59.6187°   a = 81.3508°

$$I = -\frac{\sin[(90-59.6187)+38.0513]+\sin[(90-59.6187)-38.0513]}{2}$$
$$= -\frac{\sin[(90-59.6187)+38.0513]-\sin[38.0513-(90-59.6187)]}{2} = -0.398259$$

$$II = \frac{\sin[(90-59.6187)+38.0513]-\sin[(90-59.6187)-38.0513]}{2}$$
$$= \frac{\sin[(90-59.6187)+38.0513]+\sin[38.0513-(90-59.6187)]}{2} = 0.531727$$

$$III = II \times \cos(81.3508) = 0.079963$$

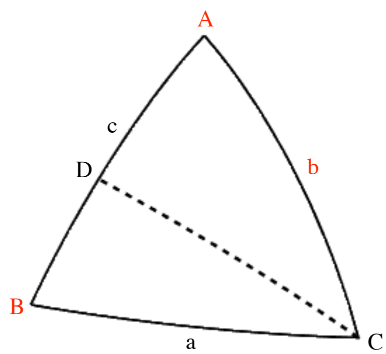
$$x = I + III = -0.318296$$

$$A = \cos^{-1}(-0.318296) = 180 - \cos^{-1}(0.318296) = 108.560°$$

Algorithm S8

Given two angles and a side opposite one in a non-right triangle, to find the remaining sides and angle.

Given A, B and b, find the rest ($\angle D = 90°$):



$$x = \sin(A)\sin(b) = \frac{\cos(A-b)-\cos(A+b)}{2}$$

$$\overset{\frown}{DC} = \sin^{-1}(x)$$

$$x = \frac{\cos(b)}{\cos(\overset{\frown}{DC})} \quad \text{; or use complements}$$

$$\overset{\frown}{AD} = \cos^{-1}(x)$$

$$x = \frac{\sin(\overset{\frown}{AD})}{\sin(b)}$$

$$\angle ACD = \sin^{-1}(x)$$

$$x = \frac{\sin(\overset{\frown}{DC})}{\sin(B)}$$

$$\checkmark\ a = \sin^{-1}(x)$$

$$x = \frac{\cos(\overset{\frown}{BC})}{\cos(\overset{\frown}{DC})} \quad \text{; or use complements}$$

$$\overset{\frown}{BD} = \cos^{-1}(x)$$

$$\checkmark\ c = \overset{\frown}{AD} + \overset{\frown}{BD}$$

$$x = \frac{\sin(\overset{\frown}{BD})}{\sin(a)}$$

$$\angle BCD = \sin^{-1}(x)$$

$$\checkmark\ C = \angle ACD + \angle BCD$$

Example S8

A = 62.9661° B = 70.3208° b = 48.6174° (D = 90°)

$$x = \frac{\cos(14.3487) - \cos(111.5835)}{2} = \frac{\cos(14.3487) + \cos(68.4165)}{2} = 0.668331$$

$$\widehat{DC} = \sin^{-1}(0.668331) = 41.9384°$$

$$x = \frac{\cos(48.6174)}{\cos(41.9384)} = 0.888716$$

$$\widehat{AD} = \cos^{-1}(0.888716) = 27.2876°$$

$$x = \frac{\sin(27.2876)}{\sin(48.6174)} = 0.611022$$

$$\angle ACD = \sin^{-1}(0.611022) = 37.6635°$$

$$x = \frac{\sin(41.9384)}{\sin(70.3208)} = 0.709787$$

$$a = \sin^{-1}(0.709787) = 45.2176°$$

$$x = \frac{\cos(45.2176)}{\cos(41.9384)} = 0.946968$$

$$\widehat{BD} = \cos^{-1}(0.946968) = 18.7432°$$

$$c = 27.2876° + 18.7432° = 46.0308°$$

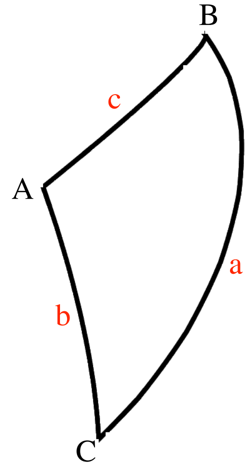
$$x = \frac{\sin(18.7432)}{\sin(45.2176)} = 0.452709$$

$$\angle BCD = \sin^{-1}(0.452709) = 26.9176°$$

$$C = 37.6635° + 26.9176° = 64.5811°$$

Algorithm S9
Given three sides of a non-right triangle, to find any angle.

Given a, b and c, find A:



$$I = \cos(b)\cos(c) \quad ;(b > c)$$
$$= \frac{\sin[(90-b)+c] + \sin[(90-b)-c]}{2}$$
$$= \frac{\sin[(90-b)+c] - \sin[c-(90-b)]}{2}$$

$$II = \sin(b)\sin(c)$$
$$= \frac{\sin[(90-b)+c] - \sin[(90-b)-c]}{2}$$
$$= \frac{\sin[(90-b)+c] + \sin[c-(90-b)]}{2}$$

$$III = \sin(90-a) - I$$
$$= \cos(a) - I$$
$$x = \frac{III}{II}$$
$$\checkmark\, A = \cos^{-1}(x)$$

Example S9
a = 94.3288° b = 40° c = 74.7178°

$$I = \frac{\sin[(90-40)+74.7178] - \sin[74.7178-(90-40)]}{2} = 0.201909$$

$$II = \frac{\sin[(90-40)+74.7178] + \sin[74.7178-(90-40)]}{2} = 0.620058$$

$$III = \cos(94.3288) - I = -0.277388$$
$$x = \frac{-0.277388}{0.620058} = -0.447358$$
$$A = \cos^{-1}(-0.447358) = 180 - \cos^{-1}(0.447358) = 116.574°$$

5 Tycho Example

The algorithms presented in this handbook describe how math was done at *Uraniborg* where the domain of application was astronomy. Therefore, it is appropriate to show an example, carried out by Tycho, in which they were used to solve a problem that he needed to solve, in this case, to determine the position of Saturn, stated as follows:

Calculus pro loco ♄ ex obſeruationibus Auguſtæ habitis Anno 1570 ultimo die Martij h. 9$\frac{1}{4}$ 4.
Diſtantia ♄ a Spica ♍ 4° 52′
Diſtantia ♄ a 5^{ta} mer. alæ ♍ 6° 25′

Calculation of the position of Saturn from observations at Augsburg at 9$\frac{1}{4}$ *hours on the last day of March in the year 1570.*
Distance of Saturn from Spica in Virgo 4°52′
Distance of Saturn from the fifth of the south wing of Virgo [θ Vir] 6°25′

In 1570, Tycho was just 23 years old[18] and *Uraniborg* would not be constructed until six years later so the data used in this calculation, obtained with a huge, new quadrant[19] that he had designed himself, were not as good as he would have eventually. [4, pg. 33] Still, this example (see Fig. 4) shows the kind of trigonometry he had to do just to make a single observation. The details were included in his logs. [3, pp. 31–32] The following pages provide a summary of the calculation.

[18] These observations were made about 2 1/2 years before Tycho observed the supernova he described in *De Nova Stella*.
[19] *Quadrans maximus*, with a radius of 5 1/2 meters

Spicæ { longit. 17 49⅙ ♎ / lat: 1 59 M. } 5^{tæ} alæ ♍ { longit. 12° 10⅓ ♎ / latit: 1 45 B. }

Ex his locis inveſtigatur locus apparens ♄ in modum ſequentem.

A polus Eccliptticæ — C ♄^{nus}
B 5ta alæ ♍ — D Spica ♍

I. In Triangulo ABC dantur BA 88 15 } compl. latit. { 5^{tæ} alæ
DA 91 59 } compl. latit. { Spicæ
BAD 5 38⅔ differentia longit.

Ergo datur BD 6° 46⅓ diſtantia ab inuicem.

II. In eodem triangulo ex datis tribus lateribus invenitur angulus ABD 123° 25′.

III. In triangulo BCD ex datis itidem tribus lateribus.
BC 6 22 } diſtantia 5^{tæ} { ♄
BD 6 46⅓ } alæ a { Spica
CD 4 48 diſtantia ♄ a Spica. Pateſcit angulus CBD 43°18′.

IV. Subducto angulo CBD ab angulo ABD relinquitur angulus ABC 80° 7′.

V. In triangulo ABC ex duobus lateribus angulum datum ambientibus innoteſcit latus reliquum CA 87 9¾ compl: lat: ♄.

VI. In triangulo ABC ex datis denique tribus lateribus datur angulus BAC 6° 19′ 40″, addendus longitudini 5^{tæ} alæ ♍ et ponitur
♄ Longit. 18° 30′ ♎, Latit. 2° 50¼ B.

Reſoluto iam calculo examinis ergo redeunt eædem ferme diſtantiæ ♄^{ni} a Spica et 5ta alæ ♍ obſeruatæ.

[Latitudo debet eße ad ſummum 2 48½ Boreal.][1].

Deinde[2] in globo magno ducta linea recta a Spica in ſtellam dextri humeri Bootis, cujus fuit Longit. 26 50 ♎, Lat. 48 40 B. ex abaco ſtellarum, et ex ea diſtantia ♄^{ni} a Spica ſubducta colligitur ♄ Longit. 18 31 ♎ Lat. 2 49 B. Quod cum priori ♄ loco fere convenit.

Denique eadem veſpera diſtabat tertia in mer. ala ♍ a ♄ 14° 15′ in eadem latitudine. Hujus longit. 4 9 0 ♎. latit. 2 49⅗ B. Addita igitur diſtantia ♄ huic longitudini provenit ἐν πλάτ. Longit. ♄ 18 26 ♎ [quod non multum diſcrepat ſed de diſtantia obſeruata non ſum ſatis certus. Si poneres diſtantiam 3^{æ} in ala et ♄ 14 20 recte conveniret][3].

Supputatio pro invenienda ☍ ♄^{ni} apparentis cum ſimplici ☉ ex primo ejus loco dato. H. 9 M. 15 Simplex ☉^{lis} 18 32¼ ♈
Longit. ♄ 18 30 ♎

Differentia 2¼ cui tempus congruit M. 45 ſubtrah. in motibus 2 circiter. Ergo ☍ ♄ cum ſimplici ☉^{lis} fuit Menſis Martij die 31 H. 8⅗ in 18° 30′ 35″ ♎ [Vide poſt vnum folium vbi hinc ex globo exactius limitavi tam quo ad longitudinem quam ad latitudinem. Noſtra præceſ. 27° 49′ 45″][4].

Ad tempus ☍ colliguntur ex tabulis Prutenicis[5].

1 In margine: »Manu Tychonis recentiori«.
2 In margine: »Longimontani«.
3 In margine adſcriptum eſt: »Manu Tychonis inſerta poſterius«.
4 In margine: »Manu Tychonis recentius addita«.
5 In margine: »Longimontani«.

Figure 4: Finding the Position of Saturn

Table 1: Ecliptic Coordinates at this Time
(with true values appended)

	Tycho		True	
	Longitude	Latitude	Longitude	Latitude
Spica	$197°49'10''$	$-1°59'00''$	$197°46'40.8''$	$-2°3'35.5''$
θ Vir	$192°10'20''$	$1°45'00''$	$192°8'9.8''$	$1°50'32.9''$
Saturn	?	?	$198°30'53.4''$	$2°47'41.9''$

From these positions,[20] the apparent position of Saturn is determined as follows:

In the figure shown, A = pole of the Ecliptic, B = θ Vir, C = Saturn and D = Spica.

I. In $\triangle ABC$ [ABD], are given $\widehat{BA} = 88°15'$, $\widehat{DA} = 91°59'$ (latitude complements) and $\angle BAD = 5°38'40''[5°38'50'']$ (longitude difference).
Therefore, separation $\widehat{BD} = 6°46'20''[6°46'08'']$.

II. In the same triangle, given three sides, is found $\angle ABD = 123°25'[123°27']$.

III. Likewise, from three sides given in $\triangle BCD$, we have the following separations:

$\widehat{BC}$ (Saturn from θ Vir) = $6°22'[6°25'$ observed]

$\widehat{BD}$ (Spica from θ Vir) = $6°46'20''$

$\widehat{CD}$ (Saturn from Spica) = $4°48'[4°52'$ observed]

Clearly, $\angle CBD = 43°18'$. [using observed separations]

IV. Subtracting $\angle CBD$ from $\angle ABD$ gives $\angle ABC = 80°7'$.

V. In $\triangle ABC$, from two surrounding sides, the given angle determines the remaining side, $\widehat{CA} = 87°9'45''[87°9'41'']$, the complement of the latitude of Saturn.

VI. Finally, in $\triangle ABC$, from three given sides, we get $\angle BAC = 6°19'40''[6°19'43'']$. Adding this to the longitude of θ Vir,[21] the position of Saturn is found to be
longitude = $198°30'$, latitude = $2°50'15''[2°50'19'']$.

I have determined from this calculation that Saturn is almost the same distance from Spica and θ Vir as was observed. (Its latitude must be at most $2°48'30''$ N.[22])

[20] Note: 17 49 1/6 ♎ [Libra] = $6 \times 30° + 17°\ 49\ 1/6'$
[21] At this time, Saturn was east of θ Vir.
[22] later, marginal notation by Tycho

References

[1] Brummelen, G. V. *Heavenly Mathematics*. Princeton University Press, 2017.

[2] Dryer, I., Ed. *Tychonis Brahe Dani Opera Omnia*, vol. I. Hauniae In Libraria Gyldendaliana, 1913.

[3] Dryer, I., Ed. *Tychonis Brahe Dani Opera Omnia*, vol. X. Hauniae In Libraria Gyldendaliana, 1923.

[4] Thoren, V. E. *Lord of Uraniborg*. Cambridge University Press, 2006.

[5] Viète, F. *Canon mathematiicus*. 1579.

[6] von Peuerbach, G., and Regiomontanus, J. *Tractatus super propositiones Ptolemaei de sinubus & chordis*. Johann Petreius, 1541.